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The Baker–Ericksen inequalities for hyperelastic models using a novel set of invariants of Hencky strain

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Abstract

We study a class of models that describes isotropic hyperelastic materials and that is defined using a novel set of invariants for the Hencky strain. For these models, we derive the appropriate form of the Baker–Ericksen inequalities. To illustrate an application, we then use this form of the Baker–Ericksen inequalities to develop a set of specific constitutive restrictions for a model of rubber-like materials proposed in [Criscione, J.C., Humphrey, J.D., Douglas, A.S., Hunter, W.C., 2000. J. Mech. Phys. Solids 48, 2445–2465].

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1. Introduction

In classical non-linear elasticity, the stored-energy function of a material modeled as hyperelastic and isotropic, like rubber, for example, can be written as a function of the principal invariants of an appropriate strain measure. This approach is mathematically elegant and by it one can solve analytically several basic and interesting boundary-value problems. On the other hand, for determining specific models from experimental data, the use of the principal invariants is problematic. For experiments typically used to collect stress-strain data, like biaxial stretching experiments, a high covariance among the stress response terms propagates experimental error. (The covariance issue is discussed in detail in Criscione, 2003.) As a consequence, definitively determining the hyperelastic response of, say, rubber from biaxial stretching data

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is difficult. Because one of the basic goals of the theory of elasticity is to model real materials, this difficulty is significant.

To address this issue, Criscione et al. (2000) developed an alternative set of invariants, based on the Hencky strain (also called logarithmic or natural strain). Though less elegant mathematically than the standard isotropic invariants, each alternative invariant admits a straightforward mechanical interpretation, which is not so for each of the standard invariants. More importantly, the alternative invariants yield models that do not suffer from covariance among the stress response terms. Criscione et al. (2000) demonstrate the use of their alternative invariants to accurately fit models from several classical data sets for biaxial-stretching experiments on rubber.

Although experimental difficulties motivated the development of this alternative set of invariants, still important is the mathematical analysis of models based on these invariants. As suggested by Wilber and Walton (2002), such analysis illuminates what a given model implies about the mechanical properties of the material. Moreover, this analysis provides additional restrictions on parameters, which can facilitate fitting models from experimental data.

In this paper we undertake a careful treatment of models based upon the alternative invariants proposed in Criscione et al. (2000). For these models, a classical constitutive restriction, the Baker–Ericksen inequalities, is considered. We derive the exact form that this constitutive restriction takes for models based on the alternative invariants.

Additionally, we consider a class of specific models, for which we derive exact conditions on the parameters in the model in order for the model to satisfy the constitutive restriction. These conditions are used to check several models fitted from data in Criscione et al. (2000). The final set of inequalities (4.8) and (4.9) is fairly complicated. (Compare these, for example, to the simple characterization of the Baker–Ericksen inequalities for, say, a neo-Hookean model.) This complexity is perhaps surprising, since the class of models is relatively simple and the inequalities are in fact not a complete characterization. However, if we take seriously the need to construct improved models of rubber-like materials and if we accept that these models should satisfy constitutive restrictions like the Baker–Ericksen inequalities, then, unavoidably, we must discover and implement complicated systems of inequalities like those derived below.

This paper is a small piece in a larger effort to develop better models of rubber-like materials. That current models be improved dramatically is essential in order to make progress in the material science and mechanics of such materials. (Our specific interest is in the mechanics of soft tissues, although the models we consider here are isotropic and hence not appropriate for many soft tissues, even under conditions for which these tissues can be modeled as hyperelastic.) We believe that progress can be made through a combination of experimental considerations, like those that motivated the work in (Criscione et al., 2000), and theoretical considerations, like those that motivated the research presented in this paper.

2. A model based on logarithmic strain

In this section, after briefly introducing notation and presenting some standard background information from non-linear elasticity, we define the set of alternative invariants for isotropic materials introduced in (Criscione et al., 2000). Using these invariants we then define the class of hyperelastic models that are the subject of our ensuing study.

We use Lin to denote the set of second-order tensors, which act as linear maps from $\mathbb{R}^3$ to $\mathbb{R}^3$. We write elements of Lin as uppercase boldface letters and elements of $\mathbb{R}^3$ as lowercase boldface letters. The value of a second-order tensor $\mathbf{T}$ at a vector $\mathbf{v}$ is written $\mathbf{Tv}$, while the composition of the second-order tensors $\mathbf{S}$ and $\mathbf{T}$ is written $\mathbf{ST}$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, we recall that the diad $\mathbf{a} \otimes \mathbf{b}$ is an element in Lin defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} := (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$, where $\mathbf{a} \cdot \mathbf{b}$ is the usual inner product of vectors. If $\{\mathbf{e}_j\}$, $\{\mathbf{f}_k\}$ are bases for $\mathbb{R}^3$, then

$\{\mathbf{e}_j \otimes \mathbf{f}_k\}$ is a basis for Lin. An inner product is defined on Lin by first defining $(\mathbf{a} \otimes \mathbf{b}):(\mathbf{c} \otimes \mathbf{d}) := (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$ and then extending the product by linearity to all second-order tensors.

The second-order tensor $\mathbf{F}$ denotes the deformation gradient associated with the deformation of an elastic body in $\mathbb{R}^3$, while $\mathbf{C} := \mathbf{F}^{\mathrm{T}}\mathbf{F}$ denotes the right Cauchy–Green strain tensor, where $\mathbf{F}^{\mathrm{T}}$ is the transpose of $\mathbf{F}$. The polar decomposition of the deformation gradient is $\mathbf{F} = \mathbf{VR}$; the left stretch tensor $\mathbf{V}$ is positive definite and symmetric.

We denote the first Piola–Kirchhoff stress by $\mathbf{T}$ and the Cauchy stress by $\mathbf{t}$. A material is hyperelastic if there is a real-valued stored-energy function whose gradient with respect to $\mathbf{F}$ is $\mathbf{T}$. Recall that for a hyperelastic material the principle of frame-indifference implies that the stored-energy function may be written as a function $\widehat{W}$ of the right Cauchy–Green strain $\mathbf{C}$. If a hyperelastic material is isotropic, then there is a 'reduced' stored-energy function $\widetilde{W} : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\widehat{W}(\mathbf{C}) = \widetilde{W}(\imath_1(\mathbf{C}), \imath_2(\mathbf{C}), \imath_3(\mathbf{C})), \tag{2.1}$$

where $\imath_1$, $\imath_2$, and $\imath_3$, the principal invariants of $\mathbf{C}$, are defined by

$$\imath_1(\mathbf{C}) := \mathbf{I} : \mathbf{C}, \quad \imath_2(\mathbf{C}) := \frac{1}{2}[(\mathbf{I} : \mathbf{C})^2 - \mathbf{I} : \mathbf{C}^2], \quad \imath_3(\mathbf{C}) := \det(\mathbf{C}). \tag{2.2}$$

We shall refer to $\imath_1$, $\imath_2$, $\imath_3$ as the standard invariants.

An alternative set of invariants to describe isotropic materials is defined in Criscione et al. (2000). To introduce these invariants—which we denote by K_1, K_2, K_3—we first recall that the logarithmic, or Hencky, strain, is $\mathbf{H} := \ln(\mathbf{V})$ (Anand, 1979a,b; Truesdell and Toupin, 1960). We define K_1 and K_2 by

$$K_1 := \mathrm{tr}(\mathbf{H}), \quad K_2 := (\mathrm{dev}(\mathbf{H} : \mathrm{dev}(\mathbf{H}))^{1/2}, \tag{2.3}$$

where for $(2.3)_2$, we recall that for a second-order tensor $\mathbf{A}$, $\mathrm{dev}(\mathbf{A}) := \mathbf{A} - (\mathbf{I}:\mathbf{A})\mathbf{I}/3$ is the deviatoric part of $\mathbf{A}$. Note that $\mathrm{tr}(\mathbf{H}) = \ln\lambda_1\lambda_2\lambda_3$, where λ_1, λ_2, λ_3 are the principal stretches of $\mathbf{F}$. Hence K_1 measures the 'amount of dilatation'. Also note that $\mathrm{dev}(\mathbf{H}) = \mathbf{0}$ if and only if $\lambda_1 = \lambda_2 = \lambda_3$. K_2 can be viewed as a measure of the magnitude of distortion. It is easily checked that $K_1 \in \mathbb{R}$ while $K_2 \in [0, \infty)$. Also, for $K_2 \neq 0$, we define the 'mode of distortion', K_3, by

$$K_3 := 3\sqrt{6}\det(\mathrm{dev}(\mathbf{H})/K_2). \tag{2.4}$$

It can be shown that $K_3 \in [-1, 1]$ and that $K_3 = 1$ in uniaxial extension, $K_3 = -1$ in uniaxial contraction, and $K_3 = 0$ for pure shear. See (Criscione et al., 2000) for a detailed introduction to these alternative invariants. In particular, see Section 10 in (Criscione et al., 2000) for a brief review of other sets of alternative invariants proposed by various researchers and how these relate to (2.3) and (2.4).

For a given deformation gradient $\mathbf{F}$, one can construct the map between the principal stretches of $\mathbf{F}$ and the alternative invariants just defined. It is readily checked that this map provides a 1–1 correspondence between the set of principal stretches and the set of alternative invariants. (There is a slight technicality that must be addressed for $K_2 = 0$; see Eqs. (2.8) and (2.9) in Criscione et al. (2000).) Because a hyperelastic material is isotropic if and only if its stored-energy can be written as a function of the principal stretches, it follows that the stored-energy of any isotropic hyperelastic material can be written as

$$W(K_1(\mathbf{H}), K_2(\mathbf{H}), K_3(\mathbf{H})). \tag{2.5}$$

Note that if the material is incompressible, so that it sustains only deformations that satisfy $\det(\mathbf{F}) = 1$, then $K_1 \equiv 0$ and (2.5) reduces to

$$W(K_2(\mathbf{H}), K_3(\mathbf{H})). \tag{2.6}$$

3. The Baker–Ericksen inequalities

In this section we recall the Baker–Ericksen inequalities and we derive, by a straight-forward computation, the form that these inequalities take when specialized to the model (2.5).

Consider a deformation $\mathbf{F} = \mathbf{VR}$ of an isotropic material. Let λ_1, λ_2, λ_3 denote the principal stretches of $\mathbf{V}$ and let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ denote the corresponding principal directions. Recall that for isotropic materials, $\mathbf{V}$ is co-axial with the Cauchy stress $\mathbf{t}$; we denote by t_i the principal stress, i.e., the eigenvalue of $\mathbf{t}$, corresponding to the principal direction $\mathbf{v}_i$. The Baker–Ericksen inequalities are

$$(t_i - t_j)(\lambda_i - \lambda_j) > 0 \quad \text{if } \lambda_i \neq \lambda_j, \tag{3.1}$$

which say that the greater principal stress occurs always in the direction of the greater principal stretch (Truesdell and Noll, 1965).

We now derive the form that these inequalities take for the model (2.5). We assume that $K_2 > 0$, which, it can be shown, is equivalent to assuming that the principal stretches are not all equal. A first step is to note that the Cauchy stress satisfies

$$J\mathbf{t} = \frac{\partial W}{\partial \mathbf{H}}, \tag{3.2}$$

where $J := \det(\mathbf{F})$; see (Alturi, 1984; Bruhns et al., 2001; Hill, 1978). From this and (2.5) follows:

$$\mathbf{t} = \frac{1}{J} W_{,1}\mathbf{I} + \frac{1}{J} W_{,2}\mathbf{\Phi} + \frac{1}{JK_2} W_{,3}\mathbf{Y} \tag{3.3}$$

for $K_2 > 0$, where

$$\mathbf{\Phi} := \frac{\text{dev}(H)}{K_2}, \quad \mathbf{Y} := 3\sqrt{6}\mathbf{\Phi}^2 - \sqrt{6}\mathbf{I} - 3K_3\mathbf{\Phi}. \tag{3.4}$$

(If $K_2 = 0$, isotropic symmetry requires that W depend on K_2 in such a way that $\mathbf{t} = J^{-1}W_{,1}\mathbf{I}$; see Section 5 in (Criscione et al., 2000).)

To find the principal stresses from (3.3), we first note that

$$\mathbf{\Phi}\mathbf{v}_a = \frac{\text{dev}(\mathbf{H})}{K_2}\mathbf{v}_a = \frac{1}{K_2}\left(\ln \lambda_a - \frac{\text{tr}(\mathbf{H})}{3}\right)\mathbf{v}_a \tag{3.5}$$

and that

$$\mathbf{Y}\mathbf{v}_a = (3\sqrt{6}\mathbf{\Phi}^2 - \sqrt{6}\mathbf{I} - 3K_3\mathbf{\Phi})\mathbf{v}_a = \left[\frac{3\sqrt{6}}{K_2^2}\left(\ln \lambda_a - \frac{\text{tr}(\mathbf{H})}{3}\right)^2 - \sqrt{6} - \frac{3K_3}{K_2}\left(\ln \lambda_a - \frac{\text{tr}(\mathbf{H})}{3}\right)\right]\mathbf{v}_a. \tag{3.6}$$

We now combine (3.3)–(3.6) to get

$$\begin{aligned}\mathbf{t}\mathbf{v}_a &= \frac{1}{J}\left[W_{,1}\mathbf{v}_a + W_{,2}\mathbf{\Phi}\mathbf{v}_a + \frac{1}{K_2}W_{,3}\mathbf{Y}\mathbf{v}_a\right] \\ &= \frac{1}{J}\left\{W_{,1} + \frac{1}{K_2}W_{,2}\left(\ln \lambda_a - \frac{\text{tr}(\mathbf{H})}{3}\right) + \frac{1}{K_2}W_{,3}\left[\frac{3\sqrt{6}}{K_2^2}\left(\ln \lambda_a - \frac{\text{tr}(\mathbf{H})}{3}\right)^2 - \sqrt{6} - \frac{3K_3}{K_2}\left(\ln \lambda_a - \frac{\text{tr}(\mathbf{H})}{3}\right)\right]\right\}\mathbf{v}_a.\end{aligned} \tag{3.7}$$

For convenience we set $\alpha_a := \ln\lambda_a - \text{tr}(\mathbf{H})/3$. By (3.7) we have

$$t_a = \frac{1}{J}\left[\left(W_{,1} - \frac{\sqrt{6}}{K_2}W_{,3}\right) + \left(\frac{1}{K_2}W_{,2} - \frac{3K_3}{K_2^2}W_{,3}\right)\alpha_a + \frac{3\sqrt{6}}{K_2^3}W_{,3}\alpha_a^2\right]. \tag{3.8}$$

Returning to (3.1), we see that

$$(t_a - t_b)(\lambda_a - \lambda_b) = \frac{1}{J}\left\{\left[\frac{1}{K_2}W_{,2} - \frac{3K_3}{K_2^2}W_{,3}\right](\alpha_a - \alpha_b) + \frac{3\sqrt{6}}{K_2^3}W_{,3}(\alpha_a^2 - \alpha_b^2)\right\}(\lambda_a - \lambda_b)$$
$$= \frac{1}{J}\left\{\left[\frac{1}{K_2}W_{,2} - \frac{3K_3}{K_2^2}W_{,3}\right] + \frac{3\sqrt{6}}{K_2^3}W_{,3}(\alpha_a + \alpha_b)\right\}(\alpha_a - \alpha_b)(\lambda_a - \lambda_b). \tag{3.9}$$

Note that

$$(\alpha_a - \alpha_b)(\lambda_a - \lambda_b) = (\ln\lambda_a - \ln\lambda_b)(\lambda_a - \lambda_b) > 0 \quad \text{for } \lambda_a \neq \lambda_b. \tag{3.10}$$

Therefore (3.1) is equivalent to

$$\frac{1}{K_2}W_{,2} - \frac{3K_3}{K_2^2}W_{,3} + \frac{3\sqrt{6}}{K_2^3}W_{,3}(\alpha_a + \alpha_b) > 0 \quad \text{for } \lambda_a \neq \lambda_b. \tag{3.11}$$

Recall that $|K_3| \leqslant 1$. We define $\theta := (-1/3)\sin^{-1}(K_3)$; note that $\theta \in [-\pi/6, \pi/6]$. One can use (2.8) and (2.9) in Criscione et al. (2000) along with some elementary algebra and trigonometry to show that

$$\alpha_1 + \alpha_2 = \ln\lambda_1 + \ln\lambda_2 - 2K_1/3 = \sqrt{2/3}K_2(\sin(\theta + 2\pi/3) + \sin(\theta)). \tag{3.12}$$

So for $a = 1$, $b = 2$, we can rewrite (3.11) as

$$K_2W_{,2} + 3[2(\sin(\theta + 2\pi/3) + \sin(\theta)) - K_3]W_{,3} > 0 \tag{3.13}$$

if $\lambda_1 \neq \lambda_2$. Upon making similar arguments from (3.11) for $a = 1$, $b = 3$ and for $a = 2$, $b = 3$, we see that the Baker–Ericksen inequalities take the form

$$K_2W_{,2} + S_i(K_3)W_{,3} > 0 \quad \text{for } i = 1, 2, 3, \tag{3.14}$$

where

$$S_1(K_3) := 3[2(\sin(\theta + 2\pi/3) + \sin(\theta)) - K_3], \tag{3.15a}$$

$$S_2(K_3) := 3[2(\sin(\theta) + \sin(\theta - 2\pi/3)) - K_3], \tag{3.15b}$$

$$S_3(K_3) := 3[2(\sin(\theta + 2\pi/3) + \sin(\theta - 2\pi/3)) - K_3]. \tag{3.15c}$$

Remark 3.16. Adding the left-hand sides of the three inequalities in (3.14) and using the identity

$$\sin(\theta + 2\pi/3) + \sin(\theta) + \sin(\theta - 2\pi/3) = 0, \tag{3.17}$$

we see that (3.14) and (3.17) imply

$$K_2W_{,2} - 3K_3W_{,3} > 0. \tag{3.18}$$

This inequality does not contain any of the functions defined in (3.15). In fact, we can produce a set of inequalities equivalent to (3.15) but without the functions S_i. Let A_1, A_2 and A_3 denote the left-hand sides of the three inequalities in (3.14). Then (3.14) and (3.17) along with several other standard trigonometric identities imply that

$$(A_1A_2 + A_1A_3 + A_2A_3)/3 = K_2^2W_{,2}^2 - 9(1 - K_3^2)W_{,3}^2 - 6K_2K_3W_{,2}W_{,3} > 0. \tag{3.19}$$

Also, one can check that

$$A_1A_2A_3 = K_2^3W_{,2}^3 - 9K_2^2W_{,2}^2W_{,3}K_3 - 27W_{,3}^2K_2W_{,2} + 27K_2W_{,2}W_{,3}^2K_3^2 + 27W_{,3}^3K_3 - 27W_{,3}^3K_3^3 > 0. \tag{3.20}$$

That the three new inequalities derived in (3.18), (3.19) and (3.20) are in fact equivalent to those in (3.14) follows from the observation that for any three real numbers a_1, a_2, and a_3, we have

$$a_1, a_2, a_3 > 0 \quad \text{if and only if} \quad a_1 + a_2 + a_3,\ \ a_1a_2 + a_1a_3 + a_2a_3,\ \ a_1a_2a_3 > 0.$$

However, the alternative set of inequalities (3.18)–(3.20) is difficult to use because of the presence of powers of the partial derivatives of the stored-energy function.

4. Application to a specific model

In the first part of this section we specialize the inequalities derived in Section 3 to incompressible materials described by models of the form

$$W(K_2, K_3) = h(K_2) + g(K_2)K_3, \tag{4.1}$$

where h and g are smooth functions. Later in this section, upon choosing specific functions for h and g, we derive explicit conditions on the parameters that appear in these functions so that the Baker–Ericksen inequalities are satisfied. Note that W in (4.1) is a special case of (2.6). The salient feature of (4.1) is its linear dependence on K_3, which is useful analytically. Moreover, as shown in Criscione et al. (2000), such models provide a sufficiently rich class within which one can construct stored-energy functions that fit experimental data well.

A special case of (4.1), namely, $h(K_2) = GK_2^2$ and $g(K_2) \equiv 0$, where G is a positive constant, was considered in (Bruhns et al., 2001). See also (Anand, 1979a,b). For this model, Bruhns et al. derive results related to the Legendre–Hadamard condition and Hill's inequality. From (4.2) below, it is immediate that, as one would expect, the model considered by Bruhns et al. satisfies the Baker–Ericksen inequalities if and only if $G > 0$.

Inserting (4.1) into the inequalities (3.14) yields

$$K_2(h'(K_2) + g'(K_2)K_3) + S_i(K_3)g(K_2) > 0 \quad \text{for } i = 1, 2, 3, \tag{4.2}$$

where each S_i is defined in (3.15).

Particular choices of i and K_3 now give further restrictions on h and g. Using (3.15), we see that $S_2(1) = S_3(1) = 0$. Hence for $K_3 = 1$ and $i = 2$ or 3, (4.2) implies that

$$K_2(h'(K_2) + g'(K_2)) > 0. \tag{4.3}$$

Likewise, for $K_3 = -1$ and $i = 1$ or 3, (4.2) implies that

$$K_2(h'(K_2) - g'(K_2)) > 0. \tag{4.4}$$

Also, for $K_3 = 0$—which, as mentioned earlier, corresponds to pure shear—and $i = 3$, (4.2) implies

$$K_2 h'(K_2) > 0. \tag{4.5}$$

Since $K_2 > 0$, it follows by combining (4.3)–(4.5) that

$$h'(K_2) > 0 \quad \text{and} \quad -h'(K_2) < g'(K_2) < h'(K_2) \tag{4.6}$$

are necessary conditions for (4.1) to satisfy the Baker–Ericksen inequalities. One can easily construct counterexamples to show that the conditions in (4.6) are not sufficient. For example, taking $h(K_2) = AK_2^2$ and $g(K_2) = BK_2^2$, with A, $B > 0$, one can check that (4.6) is satisfied for all $0 < K_2 < 2A/3B$ but that (4.2) will not hold for $i = 2$, $K_3 = 0$, and K_2 sufficiently close to $2A/3B$.

To derive more detailed results, we next consider a model of the form (4.1) with the specific choices

$$h(K_2) = A_1K_2^2 + A_2K_2^3 + A_3K_2^4, \quad g(K_2) = A_4K_2^3 + A_5K_2^4, \tag{4.7}$$

where A_1 to A_5 are material parameters. We seek conditions on these parameters that are either necessary or sufficient for the model to satisfy the Baker–Ericksen inequalities. As we shall see, pursuing even these modest goals leads to a set of surprisingly complicated inequalities, a point mentioned in Section 1.

A model of the form (4.7) was introduced in Criscione et al. (2000) and tested using data from two well-known sets of experiments on the biaxial stretching of rubber sheets (Jones and Treloar, 1975; Rivlin and Saunders, 1951). In (Criscione et al., 2000), parameter values were chosen that yield models that accurately fit the experimental data. Later, after we state conditions for the Baker–Ericksen inequalities, it will be of interest to check whether the specific parameters used in Criscione et al. (2000) satisfy the conditions we derive.

We get a set of conditions necessary for (4.1) and (4.7) to satisfy the Baker–Ericksen inequalities by substituting (4.7) into (4.6). To simplify the inequalities, we assume that each of the parameters $A_1, A_2, \ldots, A_5$ is different from 0. We restrict $K_2 \leqslant K_2^*$, where K_2^* is a positive constant, in order to consider only some physically reasonable range of stretches. Putting $(4.7)_1$ into $(4.6)_1$, dividing the resulting inequality by K_2, and using Lemma 4.12, which is given below, yield the following necessary condition:

$$-4A_1/x^* \leqslant 3A_2 \quad \text{and} \quad A_3 > -A_1/2(x^*)^2 - 3A_2/4x^*, \quad \text{or} \tag{4.8a}$$

$$-4A_1/x^* > 3A_2 \quad \text{and} \quad A_1A_3 > 9A_2^2/32. \tag{4.8b}$$

Similar manipulations based on $(4.6)_2$ and also using Lemma (4.12) yield the additional necessary condition:

$$A_4 \leqslant \frac{-4A_1}{x^*} - A_2 \quad \text{and} \quad \frac{9(A_2+A_4)^2}{32A_1} - A_3 < A_5 < \frac{3(A_2-A_4)}{4x^*} + \frac{A_1}{2(x^*)^2} + A_3, \quad \text{or} \tag{4.8c}$$

$$|A_4| < \frac{4A_1}{x^*} + A_2 \quad \text{and} \quad -\frac{3(A_2+A_4)}{4x^*} - \frac{A_1}{2(x^*)^2} - A_3 < A_5 < \frac{3(A_2-A_4)}{4x^*} + \frac{A_1}{2(x^*)^2} + A_3, \quad \text{or} \tag{4.8d}$$

$$A_4 \geqslant \frac{4A_1}{x^*} + A_2 \quad \text{and} \quad -\frac{3(A_2+A_4)}{4x^*} - \frac{A_1}{2(x^*)^2} - A_3 < A_5 < -\frac{9(A_2-A_4)^2}{32A_1} + A_3. \tag{4.8e}$$

A set of sufficient conditions, whose derivation is more lengthy and is given at the end of this section, is the following:

(i) $A_4A_5 > 0$ and $-4A_1/x^* - 3A_2 > -6|A_4|$ and

$$-3\sqrt{3}|A_5| > 9A_4^2/2A_1 - 9A_2|A_4|/2A_1 + 9A_2^2/8A_1 - 4A_3, \tag{4.9a}$$

(ii) $A_4A_5 > 0$ and $-4A_1/x^* - 3A_2 \leqslant -6|A_4|$ and

$$-3\sqrt{3}|A_5| > -6|A_4|/x^* - 2A_1/(x^*)^2 - 4A_3 - 3A_2/x^*, \tag{4.9b}$$

(iii) $A_4A_5 < 0$ and $-4A_1/x^* - 3A_2 \leqslant -6|A_4|$ and

$6|A_4|/x^* - 2A_1/(x^*)^2 - 4A_3 - 3A_2/x^* < 5|A_5|$ and

$$-6|A_4|/x^* - 2A_1/(x^*)^2 - 4A_3 - 3A_2/x^* < -7|A_5|, \tag{4.9c}$$

(iv) $A_4A_5 < 0$ and
$$- 6|A_4| < -4A_1/x^* - 3A_2 < 6|A_4| \quad \text{and}$$
$$9A_4^2/2A_1 - 9A_2|A_4|/2A_1 + 9A_2^2/8A_1 - 4A_3 < 5|A_5| \quad \text{and}$$
$$2A_1/(x^*)^2 - 4A_3 < 4|A_5|A_1/|A_4|x^* + 3|A_5|A_2/|A_4| - |A_5| \quad \text{and}$$
$$- 6|A_4|/x^* - 2A_1/(x^*)^2 - 4A_3 - 3A_2/x^* < -7|A_5|, \tag{4.9d}$$

(v) $A_4A_5 < 0$ and
$$- 4A_1/x^* - 3A_2 \geqslant 6|A_4| \quad \text{and}$$
$$9A_4^2/2A_1 - 9A_2|A_4|/2A_1 + 9A_2^2/8A_1 - 4A_3 < 5|A_5| \quad \text{and}$$
$$9A_4^2/2A_1 + 9A_2|A_4|/2A_1 - 9A_2^2/8A_1 - 4A_3 < -7|A_5|. \tag{4.9e}$$

Note that each of (4.9a)–(4.9e) by itself is sufficient.

Although the particular details of these inequalities are not important, their value can be demonstrated by checking these conditions for the specific models presented by Criscione et al. (2000). For models of the form (4.1) and (4.7), these authors derived two sets of parameter values, namely,

$$A_1 = .48, \quad A_2 = -.053, \quad A_3 = .088, \quad A_4 = .065, \quad \text{and} \quad A_5 = .039 \tag{4.10}$$

for the model based upon the data in Jones and Treloar (1975), and

$$A_1 = .47, \quad A_2 = -.08, \quad A_3 = .095, \quad A_4 = .045, \quad \text{and} \quad A_5 = .05 \tag{4.11}$$

for the model based upon the data in Rivlin and Saunders (1951).

For the model using the values in (4.10), we check the sufficient conditions (4.9). We start by observing that $A_4A_5 = .002535 > 0$, so that the relevant sufficient condition is either (4.9a) or (4.9b). To determine which applies, we sketch the graph of $x^* \mapsto -4A_1/x^* - 3A_2$ on, say, the interval $[1, 5]$. Fig. 1 indicates that if $x^* < \bar{x}$, where $\bar{x}$ is approximately 3.5, then the appropriate condition is (4.9b). One checks that the final inequality in (4.9b) is satisfied for any x^* between 0 and $\bar{x}$. On the other hand, if $x^* \geqslant \bar{x}$, then the appropriate condition is (4.9a). For this case we compute the two constants on each side of the final inequality in (4.9a) and verify that this inequality is satisfied. Therefore in either case the sufficient conditions are satisfied.

A similar procedure for the parameter values as in (4.11) shows that the sufficient conditions are satisfied for any choice of $x^* > 0$. Because both models satisfy the sufficient conditions, we know immediately that each satisfies the necessary conditions (4.8).

Before presenting the derivation of the conditions in (4.9), we note that it would be of interest to characterize properties of a more mathematical nature, like the Legendre–Hadamard condition, for the class of models we consider. However, some initial computations suggest that, because of the presence of the transcendental function in the definition of the Hencky strain, checking these properties is quite complicated for even the specific and relatively simple model (4.1) and (4.7). See (Sendova and Walton, in press) for some

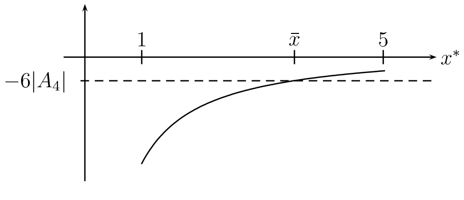

Fig. 1. $x^* \mapsto -4A_1/x^* - 3A_2$.

results in this direction. We note that in Bruhns et al. (2001), the authors successfully derive inequalities for the Legendre–Hadamard condition by restricting their attention to an important but very special case of (4.7).

Derivation of (4.9). First we present a simple lemma, which has been used several times above and will be used again below.

Lemma 4.12. *The inequality* $\alpha x^2 + \beta x + \gamma > 0$ *holds for all* $x \in (0, \bar{x}]$ *if and only if* $\gamma \geqslant 0$ *and*

$$\text{(i)}\ \ \beta \leqslant \frac{-2\gamma}{\bar{x}} \quad \textit{and} \quad \alpha\gamma > \frac{\beta^2}{4}, \quad \textit{or} \tag{4.13}$$

$$\text{(ii)}\ \ \beta > \frac{-2\gamma}{\bar{x}} \quad \textit{and} \quad \alpha > -\frac{\beta}{\bar{x}} - \frac{\gamma}{\bar{x}^2}. \tag{4.14}$$

The proof involves only elementary arguments.

As a first step to deriving (4.9), we put (4.7) into (3.14), which yields

$$2A_1K_2^2 + 3A_2K_2^3 + 4A_3K_2^4 + 3A_4K_2^3K_3 + 4A_5K_2^4K_3 + S_i(K_3)(A_4 + A_5K_2)K_2^3 > 0. \tag{4.15}$$

After cancelling K_2^2 and rearranging, we see that the problem is to find sufficient conditions on A_1 to A_5 so that

$$4A_3 + (4K_3 + S_i(K_3))A_5]K_2^2 + [3A_2 + (3K_3 + S_i(K_3))A_4]K_2 + 2A_1 > 0$$
$$\text{for all } K_3 \in [-1, 1] \quad \text{and for all } K_2 \in (0, K_2^*]. \tag{4.16}$$

Here again we are restricting $K_2 \leqslant K_2^*$, where K_2^* is a positive constant, in order to consider only some physically reasonable range of stretches. For K_3, which measures the mode rather than the magnitude of deformation, no corresponding restriction is needed.

To get started on (4.16), we set

$$B_0 = 2A_1, \quad B_1 = 3A_2 + (3K_3 + S_i(K_3))A_4, \quad B_2 = 4A_3 + (4K_3 + S_i(K_3))A_5. \tag{4.17}$$

Also, for convenience, we replace K_2 by x and K_2^* by x^*. With these changes, (4.16) can initially be viewed as the problem of finding sufficient conditions on B_1, B_2, and B_3 such that

$$B_2x^2 + B_1x + B_0 > 0 \quad \text{for all } x \in (0, x^*]. \tag{4.18}$$

First assume that $B_0 > 0$. Using Lemma (4.12) we see that (4.18) is satisfied if and only if $(B_1, B_2) \in \widetilde{\mathscr{S}}$, where $\widetilde{\mathscr{S}}$, shown in Fig. 2, is the subset of the b_1b_2-plane bounded below and to the left by the lines

$$b_2 = \frac{b_1^2}{4B_0} \quad \text{for } b_1 \leqslant -\frac{2B_0}{x^*} \quad \text{and} \quad b_2 = -\frac{b_1}{x^*} - \frac{B_0}{(x^*)^2} \quad \text{for } b_1 > -\frac{2B_0}{x^*}. \tag{4.19}$$

We can now restate (4.16) more geometrically as follows. For a fixed $A_1 > 0$, set $B_0 := 2A_1$ and seek conditions on $A_2, \ldots, A_5$ such that $(B_1, B_2) \in \widetilde{\mathscr{S}}$ for all $K_3 \in [-1, 1]$, where B_1 and B_2 are defined by $(4.17)_2$ and $(4.17)_3$.

To facilitate a geometric description of the problem, we define the curves $\mathscr{C}_i$ in the b_1b_2-plane by

$$[-1, 1] \ni \lambda \mapsto (3\lambda + S_i(\lambda), 4\lambda + S_i(\lambda)) \quad \text{for } i = 1, 2, 3, \tag{4.20}$$

and we let $\mathscr{C}$ denote the union of $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3$. Because we know the functions S_1, S_2, and S_3 explicitly, we know the curve $\mathscr{C}$ explicitly (see Fig. 3). Also, we use A_2 and A_3 to rigidly translate $\widetilde{\mathscr{S}}$ to a new set $\mathscr{S}$. Specifically, we define $\mathscr{S}$ by

$$\mathscr{S} := \{(b_1, b_2) : (b_1 + 3A_2, b_2 + 4A_3) \in \widetilde{\mathscr{S}}\}. \tag{4.21}$$

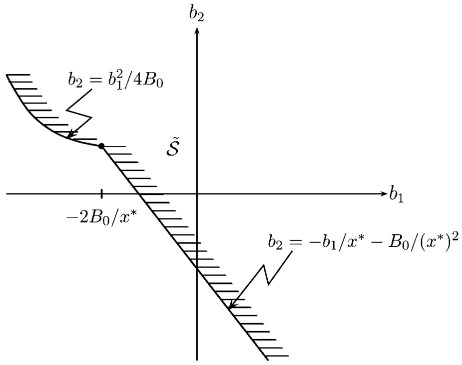

Fig. 2. $\widetilde{\mathscr{S}}$.

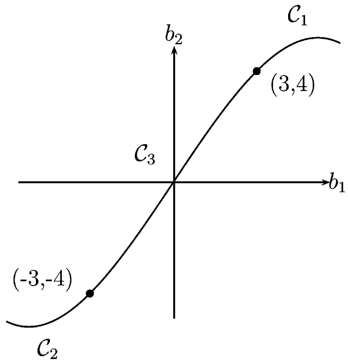

Fig. 3. $\mathscr{C}$.

By appropriately translating the boundaries of $\widetilde{\mathscr{S}}$, it can be checked that $\mathscr{S}$ is bounded below and to the left by the curves

$$b_2 = \frac{1}{8A_1}b_1^2 + \frac{3A_2}{4A_1}b_1 + \frac{9A_2^2}{8A_1} - 4A_3 \quad \text{for } b_1 \leqslant -\frac{4A_1}{x^*} - 3A_2, \tag{4.22}$$

$$b_2 = -\frac{1}{x^*}b_1 - \frac{2A_1}{(x^*)^2} - 4A_3 - \frac{3A_2}{x^*} \quad \text{for } b_1 > -\frac{4A_1}{x^*} - 3A_2. \tag{4.23}$$

Note that these two curves intersect at $(-4A_1/x^* - 3A_2, 2A_1/(x^*)^2 - 4A_3)$. Next, we denote by U the matrix

$$\mathsf{U} := \begin{pmatrix} A_4 & 0 \\ 0 & A_5 \end{pmatrix}. \tag{4.24}$$

Now we observe that the definitions of $\mathscr{C}$, $\mathscr{S}$, and U imply that the inequality (4.16) holds if and only if the image of $\mathscr{C}$ under the map $\mathbf{b} \mapsto \mathsf{U}\mathbf{b}$ is in $\mathscr{S}$. To discover conditions sufficient for this latter statement, we can first pick A_1, A_2, and A_3, construct the set $\mathscr{S}$ as above, and then seek sufficient conditions on A_4 and A_5 to ensure that the image of $\mathscr{C}$ under U is contained in $\mathscr{S}$.

Note that because **0** is always in the image of $\mathscr{C}$ under U, $\mathscr{S}$ must contain the origin. Hence we must choose A_1, A_2, and A_3 so that **0** is strictly above and to the right of the boundary of $\mathscr{S}$. From (4.22) and (4.23), we see that this yields an alternative derivation of the necessary conditions (4.8a) and (4.8b).

Now we fix A_1, A_2, and A_3 satisfying (4.8a) and (4.8b). Suppose first that both A_4 and A_5 are non-zero. Then the matrix U from (4.24) can be decomposed as

$$\mathsf{U} := \begin{pmatrix} A_4 & 0 \\ 0 & A_5 \end{pmatrix} = \begin{pmatrix} |A_4| & 0 \\ 0 & |A_5| \end{pmatrix} \begin{pmatrix} A_4/|A_4| & 0 \\ 0 & A_5/|A_5| \end{pmatrix} =: \mathsf{U}_S\mathsf{U}_R. \tag{4.25}$$

If A_4 and A_5 have the same sign, then either $\mathsf{U}_R = I$ or $-I$, where I is the 2×2 identity matrix. In either case, U_R maps $\mathscr{C}$ onto $\mathscr{C}$, and hence the task reduces to picking A_4 and A_5 such that $\mathscr{C}$, after being stretched by U_S, is contained entirely inside of $\mathscr{S}$. Consider the point $\mathbf{p} = (-6, -3\sqrt{3})$ in Fig. 4; this point is below and to the left of all the points on $\mathscr{S}$, and therefore $\mathsf{U}_S\mathbf{p} = (-6|A_4|, -3\sqrt{3}|A_5|)$ is below and to the left of all the points on the image of $\mathscr{C}$ under U_S. Observe now that if a point $\mathbf{x} \in S$, then any point above and to the right of $\mathbf{x}$ is also in $\mathscr{S}$. Hence a sufficient condition for the image of $\mathscr{C}$ under U_S to be contained in $\mathscr{S}$ is that $(-6|A_4|, -3\sqrt{3}|A_5|) \in S$, which, by (4.22) and (4.23), is satisfied if (4.9a) and (4.9b) hold.

Now suppose that A_4 and A_5 have opposite signs. Then U_R in (4.25) is a rotation by $\pi/2$, either clockwise or counterclockwise, and hence $\mathscr{C}$ gets mapped onto $\overline{\mathscr{C}}$ as shown in Fig. 5. Seeking conditions sufficient to imply that the image of $\overline{\mathscr{C}}$ under U_S is contained in $\mathscr{S}$ is more work. We could, as above, pick one point that is below and to the left of every point in $\overline{\mathscr{C}}$ and find conditions sufficient to ensure that this point is mapped into $\mathscr{S}$ by U_S. However, from Fig. 5, we see that because this point would be relatively far from $\overline{\mathscr{C}}$, these sufficient conditions would be much stronger than necessary conditions for $\overline{\mathscr{C}}$ to map into $\mathscr{S}$.

As an alternative approach, which yields sufficient conditions closer to necessary conditions, we let $\mathscr{L}$ be the line $b_2 = -b_1 - 1$ for $b_1 \in [-6, 6]$ and we let $\mathscr{L}'$ be the image of $\mathscr{L}$ under U_S. We observe that if $\mathscr{L}' \subset S$, then the image of $\overline{\mathscr{C}}$ under U_S is in $\mathscr{S}$, which is true because for every point $\mathbf{q} \in \overline{\mathscr{C}}$ there is a point on $\mathscr{L}$ that is below and to the left of $\mathbf{q}$ (see Fig. 5).

An equation for $\mathscr{L}'$ is $b_2 = -(|A_5|/|A_4|)b_1 - |A_5|$ with $b_1 \in [-6|A_4|, 6|A_4|]$. Note that the upper left point and the lower right point of $\mathscr{L}'$ are

$$\mathbf{q} := (q_1, q_2) =: (-6|A_4|, 5|A_5|), \quad \mathbf{r} := (r_1, r_2) =: (6|A_4|, -7|A_5|). \tag{4.26}$$

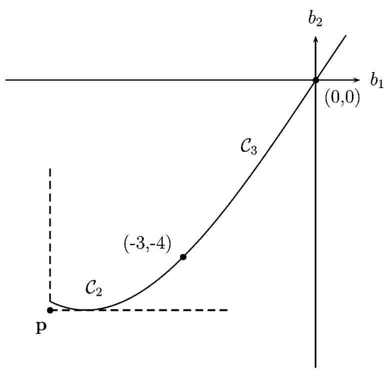

Fig. 4. Detail of $\mathscr{C}$.

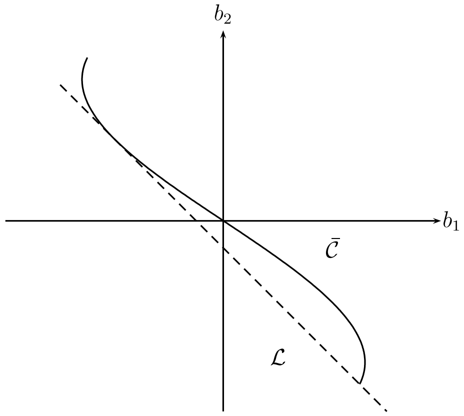

Fig. 5. $\overline{\mathscr{C}}$.

Recalling (4.22) and (4.23), we use the vertical line $b_1 = -4A_1/x^* - 3A_2$ to divide $\mathscr{S}$ into two subsets, each of which is convex. If $\mathscr{L}'$ lies entirely within one of these subsets, then $\mathscr{L}' \subset S$ if and only if $\mathbf{q}$, $\mathbf{r} \in S$. This observation yields the two inequalities (4.9c) and (4.9e), the first corresponding to $\mathscr{L}'$ lying entirely to the right of $b_1 = -4A_1/x^* - 3A_2$ and the second corresponding to $\mathscr{L}'$ lying entirely to the left of this line.

If, on the other hand, $\mathscr{L}'$ crosses the vertical line $b_1 = -4A_1/x^* - 3A_2$, then we split $\mathscr{L}'$ into the piece to the left and the piece to the right of this vertical line and we apply to each piece the same convexity argument that led to (4.9c) and (4.9e). This yields the inequalities (4.9d).

We note that one could also consider the special case where one or both of A_4 and A_5 is zero. Also, one could consider the special case where $A_0 = B_1/2 = 0$. Treating these cases by the same sort of elementary geometric arguments would yield additional inequalities similar to those derived above.

5. Conclusion

For the class of models (2.5), which describe isotropic hyperelastic materials and are defined using a novel set of invariants for the Hencky strain, we have derived the corresponding form of the Baker–Ericksen inequalities. To illustrate an application, we then used this form of the Baker–Ericksen inequalities to develop a set of specific constitutive restrictions for (4.7), a model of rubber-like materials proposed in Criscione et al. (2000). As noted in the introduction, although the model (4.7) is relatively simple, the final set of inequalities is complicated. One expects that describing constitutive restrictions like the Baker–Ericksen inequalities for other relatively simple models will generate sets of inequalities of similar complexity. Yet if we require that models should satisfy conditions like the Baker–Ericksen inequalities, then these sorts of complicated inequalities are unavoidable. We also showed that, although complex, these inequalities are readily checked for particular values of the parameters.

These results are of practical importance to the experimentalist who seeks not only to construct models that fit data well but who also wishes to understand the mechanical properties of those models according to the theory of non-linear elasticity. In the first step of the modeling process, the experimentalist typically conjectures an appropriate general form for the constitutive law of the material being modeled. This general form contains free parameters, whose values are then determined by fitting data from experiments on the material. One may start with a general model certain of whose properties depend on the values of the parameters. Of great utility both for fitting data and for assessing a model derived from data would be a set of conditions, such as inequalities on the parameters, necessary or sufficient for the final model to in fact have these properties. The inequalities we derive can serve this purpose.

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